# Continuity of the Shafer-Vovk-Ville Operator

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Monday 30 july 2018

# Discrete-time stochastic processes

## Framework

Infinite sequences of uncertain states $X_1, X_2, \ldots, X_n, \ldots$ where $X_k$ at time $k \in \mathbb{N}$ takes values in some finite set $\mathscr{X}$, called the *state space*.

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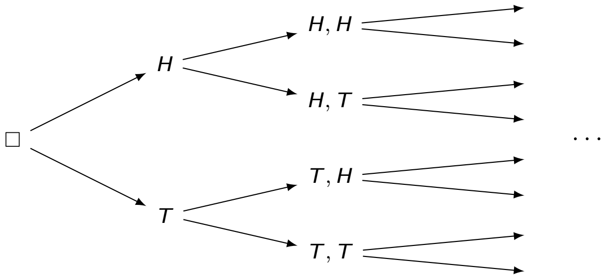
E.g. Unfair coin tossing process



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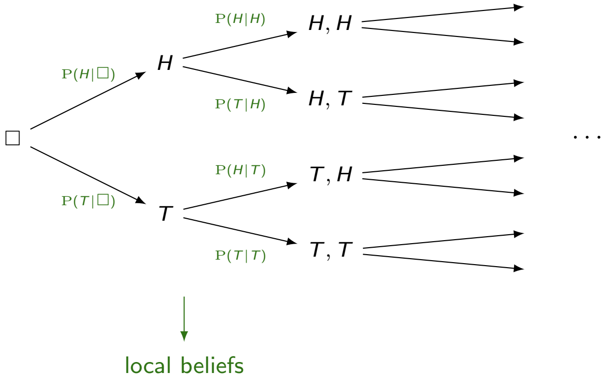

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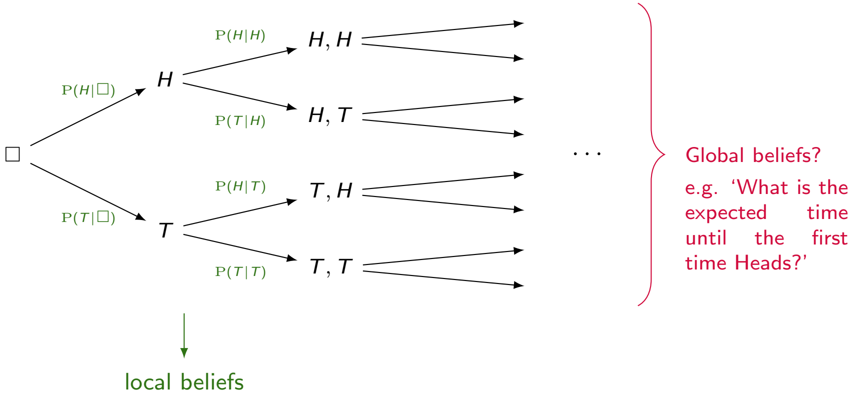

# Discrete-time stochastic processes

## Expressing global beliefs

### Kolmogorov's measure-theoretic approach

+ Elegant mathematical results
- Assumptions (e.g. measurability of gambles)
- Rather abstract, interpretation?
- Imprecise case?

# Discrete-time stochastic processes



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## Shafer and Vovk's game-theoretic approach

+ Less assumptions
+ Behavioural interpretation
+ Imprecision is naturally incorporated
− Mathematical results

# Shafer and Vovk's game-theoretic approach

## Terminology

A *situation* $x_{1:n} := (x_1, ..., x_n) \in \mathscr{X}_{1:n} := \mathscr{X}^n$ is a finite string of subsequent state values, e.g. the situation $x_{1:3} = (T, H, H)$.

A *path* $\omega$ is an infinite sequence of state values, e.g. the path 'always heads': $\omega = (H, H, H, H, \cdots)$

The set of all paths is called the *sample space* and is denoted by $\Omega := \mathscr{X}^{\mathbb{N}}$.

A *variable* $f$ is a map on the set $\Omega$ of all paths.

# Shafer and Vovk's game-theoretic approach

## Precise case

Forecaster

Skeptic

# Shafer and Vovk's game-theoretic approach

## Precise case

Forecaster

- Sets prizes $\mathrm{Q}(g|x_{1:k})$
- Sells $g^*$ for $\mathrm{Q}(g^*|x_{1:k})$
- Receives $\mathrm{Q}(g^*|x_{1:k}) - g^*$

Skeptic

- Chooses a gamble $g^*$ on $X_{k+1}$
- Buys $g^*$ for $\mathrm{Q}(g^*|x_{1:k})$
- Receives $g^* - \mathrm{Q}(g^*|x_{1:k})$

# Shafer and Vovk's game-theoretic approach

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*A martingale $\mathscr{M}$ is a gambling strategy for Skeptic.*

It associates a real number $\mathscr{M}(s) \in \mathbb{R}$ with every situation $s \in \mathscr{X}^*$.
The process difference $\Delta\mathscr{M}(x_{1:n}) \in \mathbb{G}(\mathscr{X})$, defined by

$$\Delta\mathscr{M}(x_{1:n})(x_{n+1}) \coloneqq \mathscr{M}(x_{1:n+1}) - \mathscr{M}(x_{1:n}) \text{ for all } x_{n+1} \in \mathscr{X},$$

has nonpositive expectation: $\mathrm{Q}(\Delta\mathscr{M}(x_{1:n})|x_{1:n}) \leq 0$.

If local models are imprecise: $\overline{\mathrm{Q}}(\Delta\mathscr{M}(x_{1:n})|x_{1:n}) \leq 0$.

# Shafer and Vovk's game-theoretic approach

**Definition**

$$\overline{\mathrm{E}}_{\mathrm{V}}(f) \coloneqq \inf \left\{ \mathscr{M}(\square) \colon \mathscr{M} \in \overline{\mathbb{M}}_{\mathrm{b}} \text{ and } (\forall \omega \in \Omega) \liminf \mathscr{M}(\omega) \geq f(\omega) \right\}$$

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*'The upper expectation of a variable f is the infimum starting capital such that, by gambling in the right way, we are sure to end up with a higher capital than if we would commit to the gamble f.'*

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# Mathematical results

**Example**

How can we calculate $\overline{\mathrm{E}}_{\mathrm{V}}(f)$ if $f$ is 'the time until the first time heads':

$$f(\omega) := \inf\left\{k \in \mathbb{N} \colon \omega_k = H\right\} \text{ for all } \omega \in \Omega?$$

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We can only calculate variables that depend on a finite number of states *(= n-measurable gambles)*.

In measure theory: we can use the dominated convergence of the Lebesgue integral.

*Do we have something similar for* $\overline{\mathrm{E}}_{\mathrm{V}}$*?*

# Mathematical results for $\overline{\mathrm{E}}_{\mathrm{V}}$

[Shafer G., Vovk V.: *Probability and Finance. It's Only a Game!*]

[De Cooman G., De Bock J., Lopatatzidis S.: *Imprecise stochastic processes in discrete time: global models, imprecise Markov chains, and ergodic theorems.*]

$\Rightarrow$ The restriction of $\overline{\mathrm{E}}_{\mathrm{V}}$ to the $\mathbb{G}(\Omega)$ of all bounded real-valued variables, satisfies the *coherence axioms*

E1. $\overline{\mathrm{E}}_{\mathrm{V}}(f) \leq \sup f$ for all $f \in \mathbb{G}(\mathscr{Y})$;

E2. $\overline{\mathrm{E}}_{\mathrm{V}}(f+g) \leq \overline{\mathrm{E}}_{\mathrm{V}}(f) + \overline{\mathrm{E}}_{\mathrm{V}}(g)$ for all $f, g \in \mathbb{G}(\mathscr{Y})$;

E3. $\overline{\mathrm{E}}_{\mathrm{V}}(\lambda f) = \lambda \overline{\mathrm{E}}_{\mathrm{V}}(f)$ for all $f \in \mathbb{G}(\mathscr{Y})$ and real $\lambda \geq 0$.

# Mathematical results for $\overline{\mathrm{E}}_{\mathrm{V}}$

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[Walley P.: *Statistical Reasoning with Imprecise Probabilities.*]

$\Rightarrow$ $\overline{\mathrm{E}}_{\mathrm{V}}$ is continuous with respect to *uniform convergence*

$$\lim_{n \to +\infty} \sup |f - f_n| = 0 \Rightarrow \lim_{n \to +\infty} |\overline{\mathrm{E}}_{\mathrm{V}}(f) - \overline{\mathrm{E}}_{\mathrm{V}}(f_n)| = 0$$

# Mathematical results for $\overline{\mathrm{E}}_{\mathrm{V}}$

*Is $\overline{\mathrm{E}}_{\mathrm{V}}$ continuous with respect to pointwise convergence?*

# Mathematical results for $\overline{\mathrm{E}}_{\mathrm{V}}$

*Is $\overline{\mathrm{E}}_{\mathrm{V}}$ continuous with respect to pointwise convergence?* No.

## Counterexample

$\overline{\mathrm{Q}}(h|x_{1:n}) = \max h$ for all $h \in \mathbb{G}(\mathscr{X})$ and $x_{1:n} \in \mathscr{X}^*$ (Vacuous models)

$$\Rightarrow \overline{\mathrm{E}}_{\mathrm{V}}(f) = \sup f \text{ for all } f \in \overline{\mathbb{V}}$$

Consider the events

$$A_n := \{\omega \in \Omega \colon \omega_i = H \text{ for all } 1 \leq i \leq n\} \setminus \{(H, H, H, \ldots)\}.$$

$$\Rightarrow \lim_{n \to +\infty} \mathbb{I}_{A_n} = 0 \;\Rightarrow\; \overline{\mathrm{E}}_{\mathrm{V}}(\lim_{n \to +\infty} \mathbb{I}_{A_n}) = 0$$

However, $\overline{\mathrm{E}}_{\mathrm{V}}(\mathbb{I}_{A_n}) = 1$ for all $n \in \mathbb{N}_0$

$$\Rightarrow \lim_{n \to +\infty} \overline{\mathrm{E}}_{\mathrm{V}}(\mathbb{I}_{A_n}) = 1$$

# Mathematical results for $\overline{\mathrm{E}}_{\mathrm{V}}$

**Theorem (Continuity with respect to upward convergence)**

*Consider any non-decreasing sequence of extended real variables* $\{f_n\}_{n\in\mathbb{N}_0}$ *that is uniformly bounded below—i.e. there is an* $M \in \mathbb{R}$ *such that* $f_n \geq M$ *for all* $n \in \mathbb{N}_0$*—and any extended real variable* $f \in \overline{\mathbb{V}}$ *such that* $\lim_{n\to+\infty} f_n = f$ *pointwise. If moreover* $\overline{\mathrm{E}}_{\mathrm{V}}(f) < +\infty$*, then*

$$\overline{\mathrm{E}}_{\mathrm{V}}(f) = \lim_{n\to+\infty} \overline{\mathrm{E}}_{\mathrm{V}}(f_n).$$

# Mathematical results for $\overline{\mathrm{E}}_{\mathrm{V}}$

## Theorem (Continuity with respect to cuts)

*Consider any extended real variable $f \in \overline{\mathbb{V}}$ and, for any $A, B \in \mathbb{R}$ such that $B \geq A$, the gamble $f_{(A,B)}$, defined by*

$$f_{(A,B)}(\omega) := \begin{cases} B \text{ if } f(\omega) > B; \\ f(\omega) \text{ if } B \geq f(\omega) \geq A; \\ A \text{ if } f(\omega) < A, \end{cases} \quad \text{for all } \omega \in \Omega.$$

*If $\overline{\mathrm{E}}_{\mathrm{V}}(f) < +\infty$, then*

$$\lim_{A \to -\infty} \lim_{B \to +\infty} \overline{\mathrm{E}}_{\mathrm{V}}(f_{(A,B)}) = \overline{\mathrm{E}}_{\mathrm{V}}(f).$$

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$$\lim_{A \to -\infty} \lim_{B \to +\infty} \overline{\mathrm{E}}_{\mathrm{V}}(f_{(A,B)}) = \overline{\mathrm{E}}_{\mathrm{V}}(f).$$

+ allows us to limit ourselves, for the larger part, to the study of $\overline{\mathrm{E}}_{\mathrm{V}}$ on bounded real-valued variables
+ allows for a constructive method to compute $\overline{\mathrm{E}}_{\mathrm{V}}$ for extended-real valued variables.

# Issue

Condition of $\overline{\mathrm{E}}_{\mathrm{V}}(f) < +\infty \rightarrow$ Annoying!

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**Example**

Suppose $\overline{\mathrm{Q}}(h|\square) = h(H)$ and $\overline{\mathrm{Q}}(h|s) = \max h$ for all $h \in \mathbb{G}(\mathscr{X})$ and all $s \sqsupseteq (T)$, and consider the sequence of variables

$$f_n(\omega) = \begin{cases} n \text{ if } \omega \in \Gamma(T); \\ 0 \text{ otherwise.} \end{cases}$$

$$\overline{\mathrm{E}}_{\mathrm{V}}(f_n) = 0 \text{ for all } n \in \mathbb{N}_0 \quad \leftrightarrow \quad \overline{\mathrm{E}}_{\mathrm{V}}(\lim_{n \to +\infty} f_n) = +\infty$$

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*Alternative: Use extended real-valued (super)martingales.*

→ *Interpretation???*

# Generality of $\overline{\mathrm{E}}_{\mathrm{V}}$

We know that $\overline{\mathrm{E}}_{\mathrm{V}}$ satisfies

- Local models (*n*-measurable gambles)
- Coherence on $\mathbb{G}(\Omega)$
- Continuity w.r.t. increasing sequences of variables that are uniformly bounded below and $\overline{\mathrm{E}}_{\mathrm{V}}(f) < +\infty$.

Claim: $\overline{\mathrm{E}}_{\mathrm{V}}$ on $\overline{\mathbb{V}}$ is the natural extension under these conditions!

*What properties does $\overline{\mathrm{E}}_{\mathrm{V}}$ have if you use extended real-valued supermartingales?*

# Further questions

- *'How does* $\overline{\mathrm{E}}_{\mathrm{V}}$ *relate to the measure-theoretic Lebesgue integral?'*
  $\Rightarrow$ Strong analogy in precise case.
- *'Is* $\overline{\mathrm{E}}_{\mathrm{V}}$ *an upper envelope of precise* $\overline{\mathrm{E}}_{\mathrm{V}}$*'s?'*
  $\Rightarrow$ We suspect so for limits of *n*-measurable gambles.
- *'What are the continuity properties of* $\overline{\mathrm{E}}_{\mathrm{V}}$ *when we have convergence in probability?'*
- ...

Questions?